On the teaching and learning of logic in mathematical contents

Kyeong Hah Roh
Arizona State University
khroh@asu.edu

School of Mathematical and Statistical Sciences
Arizona State University

Students' understanding of the formal definitions of limit

teaching and learning of logic in mathematical contents

My Journey in Math Education Research

Why Logic & Proof in Mathematics?

- Our society expects people to have ability to make decisions in their workplaces more efficiently by deducing valid inferences from a tremendous amount of information and resources.
- A person’s logical thinking in workplace plays a crucial role in
    - Making valid arguments from the given information
    - Evaluating the validity of others’ arguments
- Training our students as logical thinkers is a central component in education (NCTM, 2000; NGAC & CCSSO, 2010; NRC, 2005)
- Many universities offer mathematics courses to introduce logic and various proof structures for valid arguments in mathematical contexts (David & Zazkis, 2017).

A sequence $\{a_n\}$ converges to L if for any $\varepsilon > 0$, there exists a positive integer N such that for all $n > N$, $|a_n - L| < \varepsilon$.

A calculus student's interpretation of the $\varepsilon - N$ definition

Emma: Okay, umm, say you can randomly pick an integer N…, and I chose to pick 10, which means the tenth term of the sequence. And if my sequence is defined by … $\{1 - 1/n\}_{n=1}^{\infty}$, then for the 10th term, or the N-th term, a_N is equal to $1 - 1/10$, which equals 9/10. And to calculate the … ε, for that, we could see the value of a_N minus the limit which we said was 1 …. And we will take the absolute value of that difference $[9/10 - 1]$, which is equal to 1/10. And we will see that [1/10] is equal to ε…. And then we are saying that the difference of the value of any term that is after 10[th] minus the limit of the sequence will be less than the value of 1/10 because each term value $[1 - 1/n]$ is getting closer and closer to 1.

Roh, K., & Lee, Y. (2011). The Mayan activity: A way of teaching multiple quantifications in logical contexts. *Problems, Resources, and Issues in Mathematics Undergraduate Studies*, *21*(*8*), 1-14

Why do we tend to reverse the order of ε and N from the definition?

Our intuition suggests a “dynamic” idea of a limit as the result of a process of “motion”; **we move on through the row of integers 1, 2, 3, …, *n*, … and then observe the behavior of the sequence a_n.** […] we feel that the approach $a_n \to a$ should be observable. But […] to arrive at a precise definition we must reverse the order of steps; instead **of first looking at the independent variable *n* and then at the dependent variable $a_n \to a$**, we must base our definition on what we have to do if we wish actually to check the statement . In such a procedure we must first choose an arbitrarily small margin around *a* and then determine whether we can meet this condition by taking the independent variable *n* sufficiently large (Courant & Robbins, 1996, 2nd ed., p.292).

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Study 1: Clinical Interviews with Calculus Students

Roh, K. (2010). An empirical study of students' understanding of a logical structure in the definition of the limit of a sequence via the ε-strip activity. *Educational Studies in Mathematics*, *73*, 263-279.

The ε-strips are …

- made of translucent paper
  - The graph of a sequence can be observed through ε-strips.
- in the center of which a red line is drawn
  - An anticipated value for the limit of a sequence can be marked by the center line of each ε-strip.
- with constant width
  - Each error bound $\boldsymbol{\varepsilon}$ *can be* specified by the half of the width of an ε-strip.
- with indefinite length

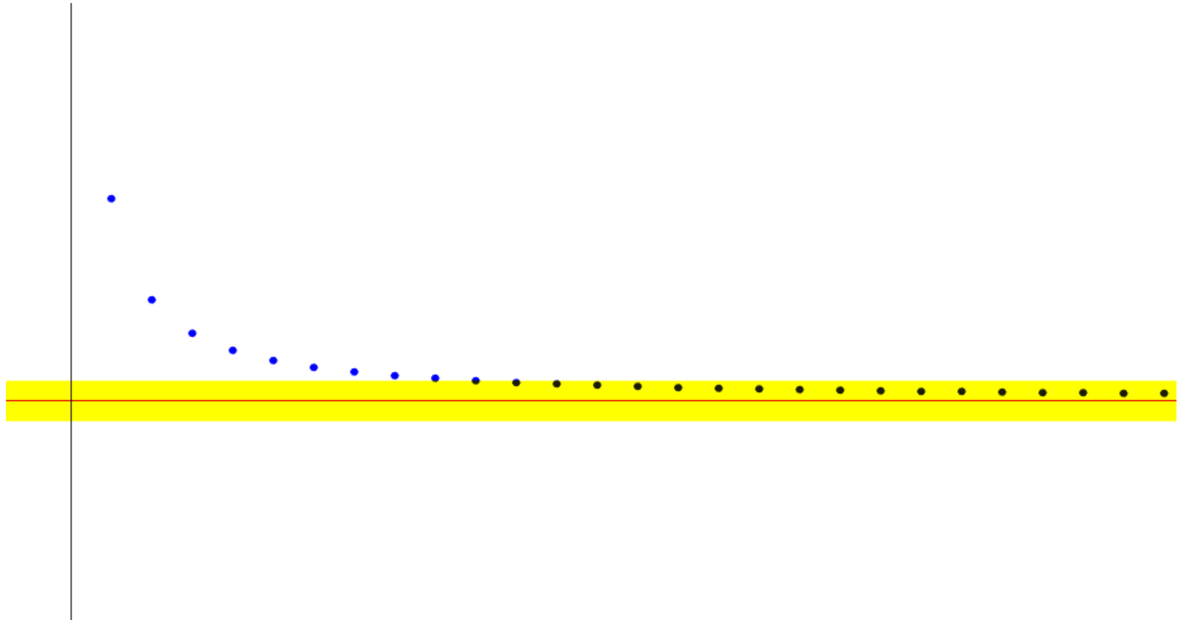
Evaluate ε-strip definitions A & B

ε-strip definition A

L is a limit of a sequence when for any *ε*-strip centered at *L*, infinitely many points on the graph of the sequence are inside the *ε*-strip.

ε-strip definition B

L is a limit of a sequence when for any *ε*-strip centered at *L*, only finitely many points on the graph of the sequence are outside the *ε*-strip .



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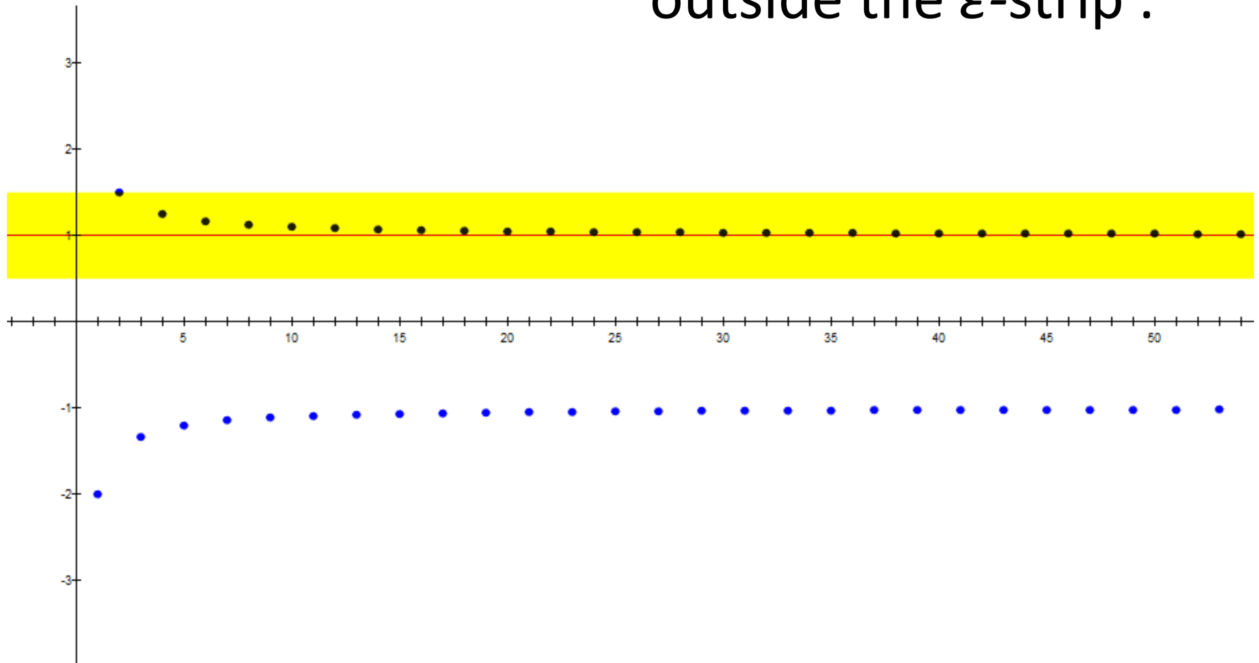


Findings: Various ways that calculus students interpret "for any $\varepsilon > 0$" in the definition

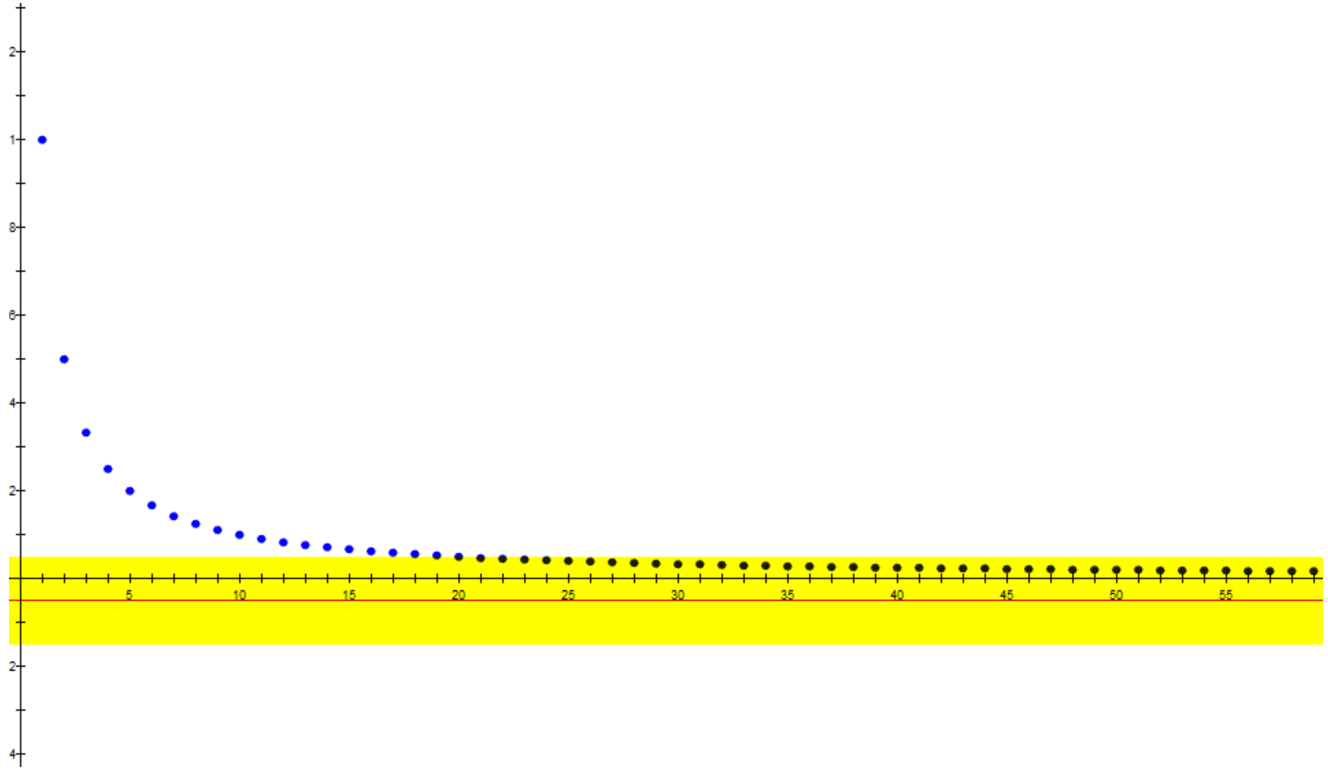
The $\varepsilon - N$ definition:

A sequence $\{a_n\}$ of real numbers is said to converge to a real number L if **for any $\boldsymbol{\varepsilon} > 0$**, there exists $N \in \mathbb{N}$ such that for all $n > N$, $|a_n - L| < \varepsilon$.

- No meaning
- 0 for a value for epsilon (single value)
- **Some positive numbers for epsilon** (a finite number of values, static)
- Any, but fixed, positive number for epsilon (an infinite number of values, static)
- A sequence of any positive numbers, decreasing towards zero (an infinite number of values, dynamic)

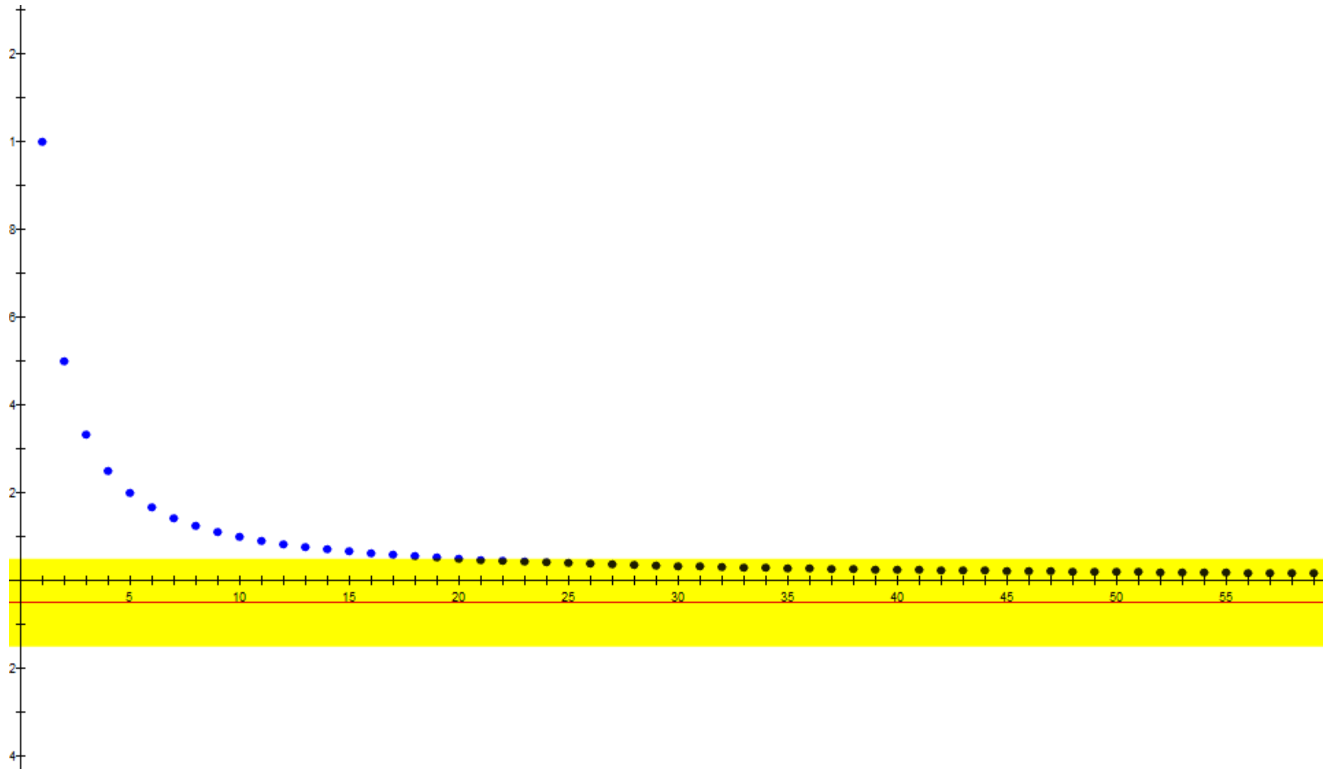
Ben’s argument

Infinitely many points on the graph of the sequence $\{1/n\}$ are inside the ε-strip when the ε-strip is centered at $y=-0.05$. Hence, accepting ε-strip definition A as a definition of limit, we should determine the value -0.05 as a limit of $\{1/n\}$.



Emma’s argument

Only finitely many points on the graph of the sequence $\{1/n\}$ are outside the ε-strip when the ε-strip is centered at $y=-0.05$. Hence, accepting ε-strip definition B as a definition of limit, we should determine the value -0.05 as a limit of the sequence $\{1/n\}$.



Study 2: Classroom Teaching Experiment in advanced calculus

Roh, K., & Lee, Y. (2017). Designing tasks of introductory real analysis to bridge a gap between students' intuition and mathematical rigor: The case of the convergence of a sequence. *International Journal of Research on Undergraduate Mathematics Education, 3*, 34-68.

The Classroom Setting

Advanced Calculus

- Proofs & Rigorous Definitions
- Math or Secondary Math Education Students

No Textbook

- Worksheets provided in class
- Class-note provided after topics are covered in class
  - Definitions & Theorems without proofs

Inquiry-Based Learning (IBL)

- Small groups (3~4 members per group)
- Two 75-minute classes & one 50-minute recitation per week for 15 weeks
- Students were asked to make and justify conjectures and to evaluate arguments.

The ε-strip activity in the Advanced Calculus Classroom

Determine the convergence of sequences provided

Evaluate ε-strip definitions A and B

Evaluate Ben & Emma's arguments about ε-strip definitions A and B

Re-evaluate ε-strip definitions A and B

Revise ε-strip definitions A and B if needed.

***ε-N* Definition Introduced to the class**

Compare the *ε-N* definition with ε-strip definitions A B

How did students come to accept ε-strip definition B as a definition for the limit of a sequence?

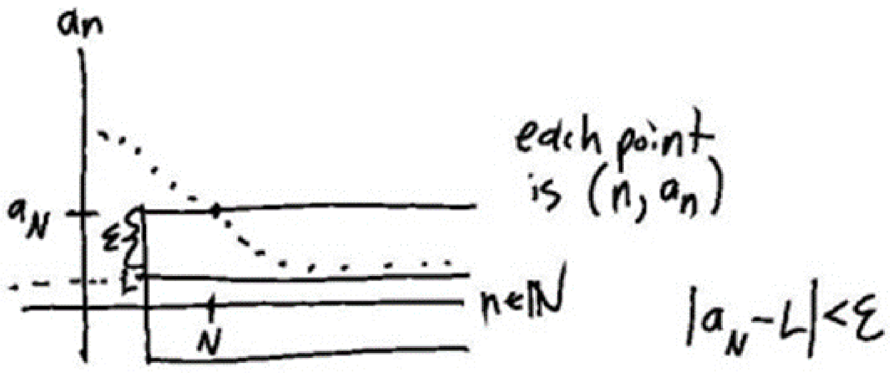
- At the beginning of the ε-strip activity, four students in Group 1 evaluated that neither ε-strip definition A nor ε-strip definition B is correct.

- Several shifts in these students' evaluations were made until they accept ε-strip definition B as a definition for the limit of a sequence:

- At the end of the ε-strip activity, they evaluated that ε-strip definition A is incorrect, and accepted ε-strip definition B.

Condition A is insufficient → Condition A is unnecessary → Condition B is necessary → Condition B is sufficient

Dave's interpretation of the $\varepsilon - N$ definition after the ε-strip activity in classroom



How I've been viewing it [ε] is a length of ε-zone. It [ε] has to be greater than 0. So for any value of ε, and this [ε] is just an arbitrary one, we can find a point *N* such that all values of small *n* that I remember are naturals after this [*N*] are contained within this value of ε. So, basically this $[a_n]$ minus the value of *L* … the absolute value of that $[|a_n - L|]$ will give you a value less than ε. […] What we find is that we can find a point where all of these points can be counted, and all these […] infinite points are contained within the ε-strip. […] Whatever arbitrary point you choose in here [inside the ε-strip] would be a_n […] as long as *n* is larger than *N.* I'm just thinking individual dots, but each of these dots will be small *n.* And if you pick any arbitrary value (*points to a dot inside the ε-strip*), then it will be a_n.

How did the ε-strip activity play a role in understanding the ε-N definition?

The ε-strip activity helped students to understand the necessity of ε *and* N

- Why ε is necessary in defining the limit of a sequence
- Why N is necessary in defining the limit of a sequence

The ε-strip activity helped students to construct dynamic mental images regarding relationships between ε and N

- Why N may depend on ε
- Why ε is independent of N (cf. Emma in Slide 6)

How did the ε-strip activity play a role in learning the next sequence of topics?

The ε-strip activity helped students to understand other definitions with the similar logical structure

- Definition of a Cauchy sequence
- Definition of the Limit of a Function
- Definition of a Continuous Function

The ε-strip activity helped students to construct proofs using definitions with the similar logical structure

- Exercise for proofs of convergence
- Proofs for theorems (e.g., Every convergent sequence is a Cauchy sequence.)

Study 3: Logical Thinking in non-mathematical contents

Roh, K., Lee, Y., & Austin, T. (2016). The King and Prisoner puzzle: A way of introducing the components of logical structures. *Problems, Resources, and Issues in Mathematics Undergraduate Studies, 26*, 424-436.

The King & Prisoner Puzzle

Once upon a time, there were a king and a prisoner. One day, the king made a suggestion to the prisoner.

King: There are two rooms in front of you. (A) In each of these rooms, there is either a beast or a key to the prison. In front of each room, there is a description about whether there is a key or a beast. (B) Only one of these descriptions is true and the other is false. If you go into a room that has a key, you will be free.

The prisoner saw the following descriptions on the door of each room:

- Room 1 (R1): In one room, there is a key; and in the other, there is a beast.
- Room 2 (R2): In this room, there is a beast; and in the next room, there is a key.

Can the prisoner be free? If the prisoner can be free, which room(s) should be selected in order for the prisoner to be free?

A Valid Solution

Suppose (R2) is true. Then there is a key in Room 1 and there is a beast in Room 2, which make (R1) true as well. However, due to (B) it is impossible for both (R1) and (R2) to be true. Hence (R2) should be false. Then by (B), (R1) must be true. Thus there are only two possible cases as follows:

> Case 1: There is a beast in Room 1 and there is a key in Room 2;
>
> Case 2: There is a key in Room 1 and there is a beast in Room 2.

Between these two cases, Case 2 makes (R2) true as well, which contradicts (B). Hence, it must be Case 1. Therefore, yes, there is a key in Room 2; hence the prisoner can be free if the prisoner chooses Room 2.

- (A) In each of these rooms, there is either a beast *or* a key to the prison.
- (B) Only one of the descriptions between R1 *and* R2 is true and the other is false.
- (R1) In one room, there is a key; *and* in the other, there is a beast.
- (R2) In this room, there is a beast; *and* in the next room, there is a key.

Student solutions to the King & Prisoner Puzzle

	Incorrect Solutions	Correct Solution	# of Students (%)
# of Students (%)	13 (28.3 %)	33 (71.7 %)	46 (100 %)

Validity of Student Solutions

	Incorrect Solutions	Correct Solution	# of Students (%)
Invalid Solutions	13 (28.3%)	31 (67.4%)	44 (95.7%)
Valid Solutions		2 (4.3 %)	2 (4.3 %)
# of Students (%)	13 (28.3 %)	33 (71.7 %)	46 (100 %)

Validity of Student Solutions in Comparison

	# of Students w/o Learning Logic	**# of Students w/ Learning Logic**
Invalid Solutions	31	13
Valid Solutions	0	2
# of Students (%)	31	15

A Student Solution to the King and Prisoner Puzzle

Because (R2) describes exactly where the key is whereas (R1) leaves the option open that either room will have a key, the prisoner might consider (R2) to be true. However, ***that is a trap that the King set for the prisoner because the king would not be happy if the prisoner is free. The king would want the prisoner to be eaten by a beast in Room 1. That means (R2) must be false.*** Then Room 1 has a beast and Room 2 has a key. Therefore, yes, there is a key in Room 2; hence the prisoner can be free if the prisoner chooses Room 2 .

Some issues in students’ logic emerged repeatedly in Problem solving

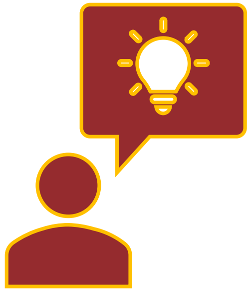
- De Morgan’s rule: used the word ‘and’ in the negations of the statements instead of replacing it to ‘or.’
- The word ‘a’ in quantification: the word ‘a’ in the phrase ‘there is a key’ or ‘there is a beast’ as a singular quantity.
- Converse error (Epp, 2003): considered improperly that (R1) implies (R2) because (R2) satisfies (R1).

Many students did not monitor the validity of their solutions.

- Some students show logical inconsistency in their arguments in the sense that they did not recognize that new claims that they generated from the given information **contradicts** with other given information.
    - Although some students recognized a contradiction in their argument, they were unable to figure out how to handle the contradiction to solve the problem.
- Some other students did not complete the validation process.

Study 4.
Students’ Cognitive Consistency in Their Logical Thinking in mathematical contents

Roh, K., & Lee, Y. (2018). Cognitive consistency and its relationships to knowledge of logical equivalence and mathematical validity. *Proceedings for 21st annual Conference on Research in Undergraduate Mathematics Education*. San Diego, CA



Cognitive Consistency (CC)

- Cognitive consistency refers to an intra-individual psychological pressure to self-organize one's beliefs and identities in a balanced fashion" (Cvencek, Meltzoff, & Kapur, 2014).
- People behave in ways that maintain cognitive consistency among interpersonal relations, intrapersonal cognitions, beliefs, feelings, or actions (Bateson, 1972, Festinger, 1957; McGuire, 1966)

Cognitive Consistency (CC) in **Logical Thinking**

- An individual psychological pressure to self-organize his/her thinking to have no logical contradiction
- People behave in ways that maintain consistency among claims in their arguments, with **no logical contradictions** among the claims .
  - From given (or available) information, a person might deduce two claims, ‘x is an integer’ and ‘x is not an integer’.
  - The person might recognize that two claims, ‘x is an integer’ and ‘x is not an integer’ contradict one another.
  - Once the person recognizes such a logical contradiction, he/she would attempt to find a way to remove it from his/her argument in order to maintain cognitive consistency in his logical thinking.

Questions for CC in Logical Thinking

	Statement Format	Truth-value	Argument Structure	Validity
Q8	Simple Statement with Two Quantifiers	True	Example/counterexample	invalid
Q9	Conditional Statement with One Universal Quantifier	True	Direct proof	invalid
Q10	Simple Statement with One Universal Quantifier	False	Proof by contradiction	valid
Q11	Simple Statement with Universal Quantifier in the premise	True	Proof by contrapositive	invalid
Q12	Simple Statement with One Universal Quantifier	True	Proof by mathematical induction	invalid

Format of the questions for CC

- Each question consists of a statement, one or two arguments about the statement, and four sub-questions:

Given a statement and an argument about the statement,

1) Determine the truth-value of a given statement (Multiple Choice)
2) Determine if the argument is to prove or to disprove the statement (Multiple Choice)
3) Evaluate the validity of the argument (Multiple Choice)
4) Explain why you think (Open ended)

Example: Q9

Q9. An integer a is said to be odd if and only if there exists $n \in \mathbb{Z}$ such that $a = 2n + 1$. Tim was asked to prove or disprove:

(♣) For any positive integers *x* and *y*, if *x* and *y* are odd, then xy is odd.

The following is Tim's argument:

$$x = 2n + 1, n \in \mathbb{Z}$$

$$y = 2n + 1, n \in \mathbb{Z}$$

Therefore, $xy = (2n + 1)(2n + 1) = 4n^2 + 4n + 1 = 2(2n^2 + 2n) + 1$ is odd.

1) Check the most appropriate one about the statement (♣).

a. The statement (♣) is true.

b. The statement (♣) is false.

c. We cannot determine if the statement (♣) is true or false.

2) Check the most appropriate one to describe what Tim attempted to prove.

a. Tim attempted to prove the statement (♣) is true.

b. Tim attempted to prove statement (♣) is false.

c. We cannot determine if Tim attempted to prove the statement (♣) is true or he attempted to prove the statement (♣) is false.

3) Check the most appropriate one to describe if Tim's argument is valid.

a. Tim's argument is valid as a proof of the statement (♣).

b. Tim's argument is invalid as a proof of the statement (♣).

c. We cannot determine if Tim's argument is valid or invalid.

Scoring Rubric for Cognitive Consistency (CC) in Logical Thinking

A Student's Score of Cognitive Consistency (CC)

$= \sum_{i=8}^{12}(\text{CC score from } Q_i)$

- Each question was scored either -1 (cognitive inconsistency) or 0 (no cognitive inconsistency).

- CC scores could be possibly ranged from -5 to 0.

Example of Cognitive Consistency (Q9)

Examples of Cognitive Consistencies		
(1) Is the given statement is true?	(2) What does the given argument attempts to?	(3) Is the given argument is valid?
(a) True	**(b) Prove False**	**(b) Invalid**
(b) False	(c) Prove True	(b) Invalid
(a) True	(b) Prove True	(a) Valid

Q9. An integer *a* is said to be odd if and only if there exists $n \in \mathbb{Z}$ such that $a = 2n + 1$.
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CC score on Q9 = 0

Examples of Cognitive Inconsistency (Q9)

Cognitive Inconsistencies in Logical Thinking		
(1) Is the given statement is true?	(2) What does the given argument attempts to?	(3) Is the given argument is valid?
(a) **True** or (c) Cannot determine	(b) **Prove False**	(a) **Valid**
(b) False or (c) Cannot determine	(c) Prove True	(a) Valid

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CC score on Q9 = -1

Examples of Cognitive Inconsistency (Q9)

Cognitive Inconsistencies in Logical Thinking		
(1) Is the given statement is true?	(2) What does the given argument attempts to?	(3) Is the given argument is valid?
(a) True or (c) Cannot determine	(b) Prove False	(a) Valid
(b) **False** or (c) Cannot determine	(c) **Prove True**	(a) **Valid**

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(♣) For any positive integers x and y, if x and y are odd, then xy is odd.

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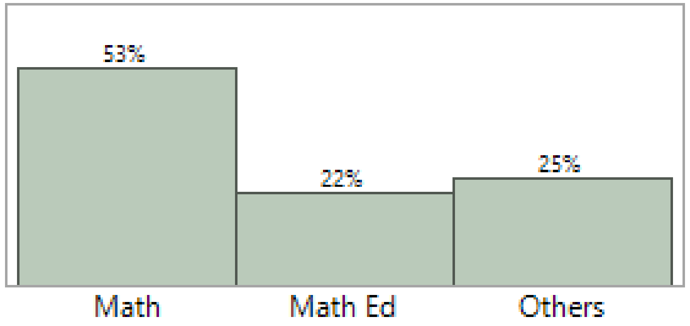
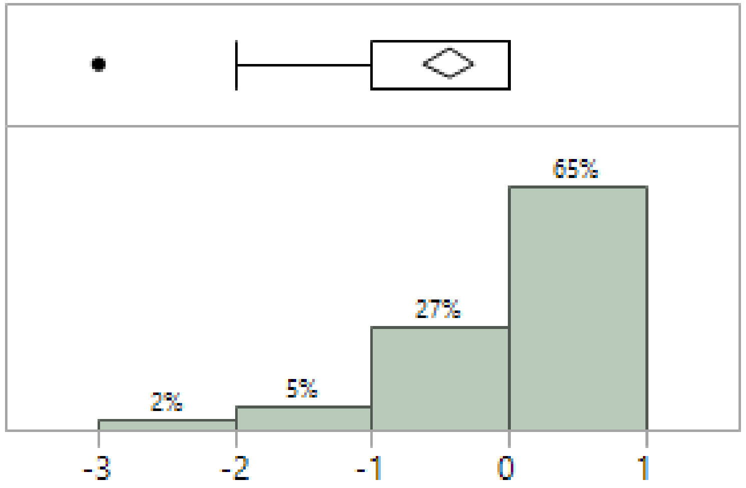
Major-pre



Frequencies

Level	Count	Prob
Math	29	0.52727
Math Ed	12	0.21818
Others	14	0.25455
Total	55	1.00000

N Missing 0
3 Levels



Quantiles

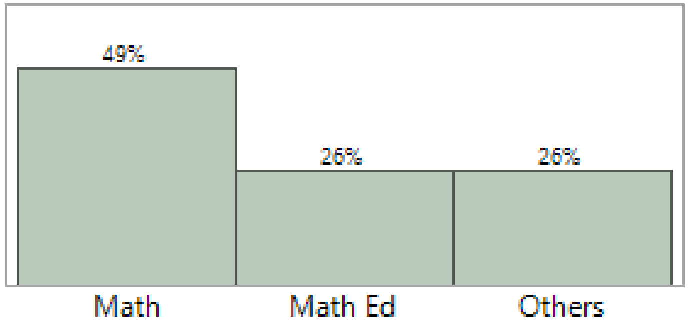
100.0%	maximum	0
99.5%		0
97.5%		0
90.0%		0
75.0%	quartile	0
50.0%	median	0
25.0%	quartile	-1
10.0%		-1
2.5%		-2.6
0.5%		-3
0.0%	minimum	-3

Summary Statistics

Mean	-0.436364
Std Dev	0.6875517
Std Err Mean	0.0927094
Upper 95% Mean	-0.250492
Lower 95% Mean	-0.622235
N	55

Pretest Results (N=55)

Major-post

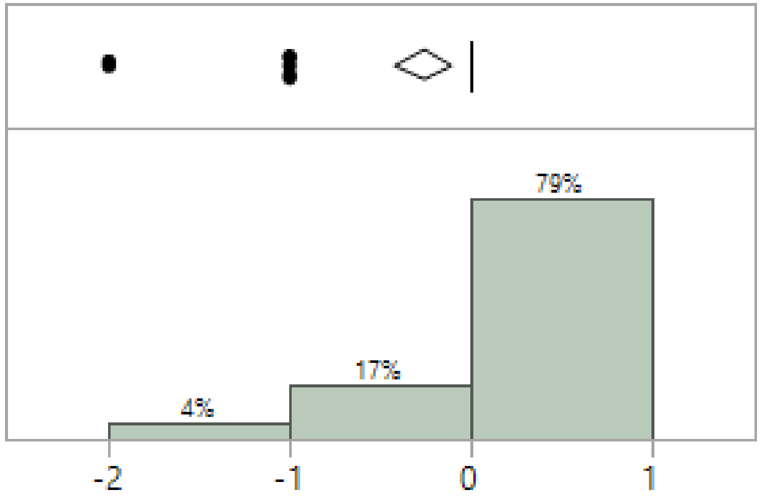


Frequencies

Level	Count	Prob
Math	23	0.48936
Math Ed	12	0.25532
Others	12	0.25532
Total	47	1.00000

N Missing 0

3 Levels



Quantiles

100.0%	maximum	0
99.5%		0
97.5%		0
90.0%		0
75.0%	quartile	0
50.0%	median	0
25.0%	quartile	0
10.0%		-1
2.5%		-2
0.5%		-2
0.0%	minimum	-2

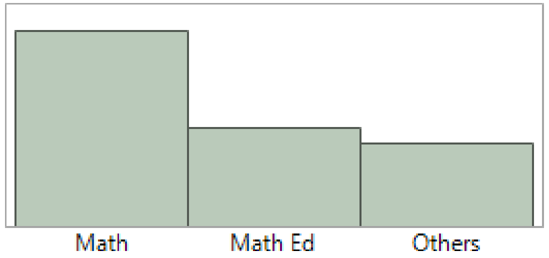
Summary Statistics

Mean	-0.255319
Std Dev	0.5303028
Std Err Mean	0.0773526
Upper 95% Mean	-0.099616
Lower 95% Mean	-0.411022
N	47

Posttest Results (N=47)

Posttest – Pretest Comparison

Major (pre)

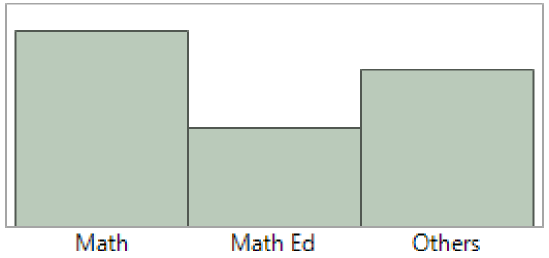


Frequencies

Level	Count	Prob
Math	12	0.52174
Math Ed	6	0.26087
Others	5	0.21739
Total	23	1.00000

N Missing 0
3 Levels

Major (post)

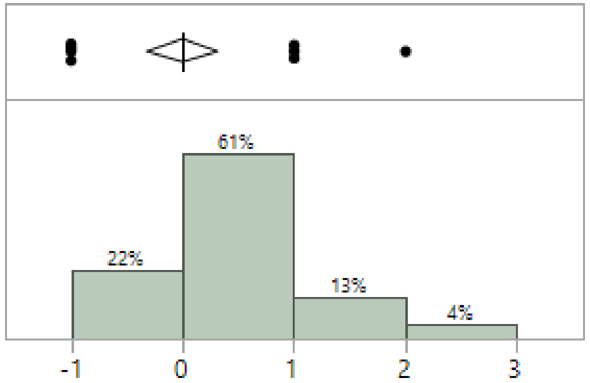


Frequencies

Level	Count	Prob
Math	10	0.43478
Math Ed	5	0.21739
Others	8	0.34783
Total	23	1.00000

N Missing 0
3 Levels

CC (post-pre)



Quantiles

100.0%	maximum	2
99.5%		2
97.5%		2
90.0%		1
75.0%	quartile	0
50.0%	median	0
25.0%	quartile	0
10.0%		-1
2.5%		-1
0.5%		-1
0.0%	minimum	-1

Summary Statistics

Mean	0
Std Dev	0.7385489
Std Err Mean	0.1539981
Upper 95% Mean	0.3193725
Lower 95% Mean	-0.319373
N	23

Some issues with Transition-to-Proof courses

- Debates have been made whether or not students can learn mathematical logic when it was presented in mathematical contents.
- However, cognitive consistency in logical thinking might NOT have been treated as a crucial component in transition-to-proof courses.
- Designing tasks or instructional interventions would be needed to help students recognize cognitive inconsistencies in their logical thinking if they have any.

Suggestions for classroom activities

- Ask students to (1) determine the truth-value of the statements before proving or disproving the statements
- Add more activities/tasks for the validation of someone else' arguments. In the validation activities, ask students

  (2) Determine if the argument is to prove or to disprove the statement

  (3) Evaluate the validity of the argument
- Facilitate student discussion even after proof construction or proof validation activities

Direction for Future Research

- Refining the Mathematical Logic Instrument (MLI) to include
  - Various types of statements
  - Various types of arguments to be paired with the statements
- Scale-up the cognitive consistency study (Roh & Lee, 2018) via the revised MLI
  - Investigation of the correlation between cognitive consistency in students' logical thinking and their knowledge of mathematics, mathematical logic, & mathematical proofs.
- Design of instructional interventions for cognitive consistency in logical thinking

Thank you!

School of Mathematical and Statistical Sciences
Arizona State University

Kyeong Hah Roh
khroh@asu.edu